

# Generalized Migdal-Kadanoff Bond-moving Renormalization Recursion Procedure I: Symmetrical Half-length Bond Operation on Translational Invariant Lattices

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## Abstract

We report in a series of papers two types of generalized Migdal-Kadanoff bond-moving renormalization group transformation recursion procedures. In this first part the symmetrical operation of half length bonds on translational invariant lattices are considered. As an illustration of their predominance in application, the procedures are used to study the critical behavior of the spin-continuous Gaussian model constructed on the triangular lattices. Results such as the correlation length critical exponents obtained by this means are found to be in good conformity with the classical results from other studies.

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## I. INTRODUCTION

In the early 1970s a very important contribution to the calculation of near-critical properties in many-particle systems was made by A. A. Migdal in which a set of renormalization group recursion formulas were proposed basing on a series of bond-moving operations [1, 2]. Soon after its publication, L. P. Kadanoff consummated Migdal's remarkable work to include potential-moving operations [3]. This enables the bond-moving procedure together with the decimation [4, 5] and block transformation [6–11] to form three pillars of coarsening methods in the study of phase transition and critical phenomena via renormalization group theory.

The bond-moving procedure is important because it can be applied to systems with almost all kinds of internal symmetries. The results got from them are asymptotically exact and can be analytically continued in dimensionality with relative ease in the limits of both weak and strong couplings [12, 13]. With these advantages, it has offered the possibility of attacking strong-coupling phase transitions in giving an exact determination of the critical point of the two-dimensional Ising system with dual symmetries [14, 15] and has been an effective starting point for dealing with highly anisotropic systems such as the Heisenberg model at dimensionality  $2 + \epsilon$ .

Effective as it has always been, the Migdal-Kadanoff bond-moving recursion procedure is recently found not as consummate as it should have been. For example, it is preferred in dealing with spin-discrete systems constituted on traditional globally symmetric lattices. For systems with local symmetries such as some fractals it appears to be sometimes inconvenient or even powerless. For these reasons, we aim in this paper to improve the Migdal-Kadanoff bond-moving recursion procedure to seek a more expanded rang of its application.

The paper is organized as follows: in Sec. II, we give the improvement we have made on the procedures by recurring it on the triangular lattices; in Sec. III, the critical behavior of the spin-continuous Gaussian model is studied by using of the newly generalized procedures as an illustration of their predominance in application. Sec. IV serves as a summary of our conclusion in which further implicit applications are also discussed.

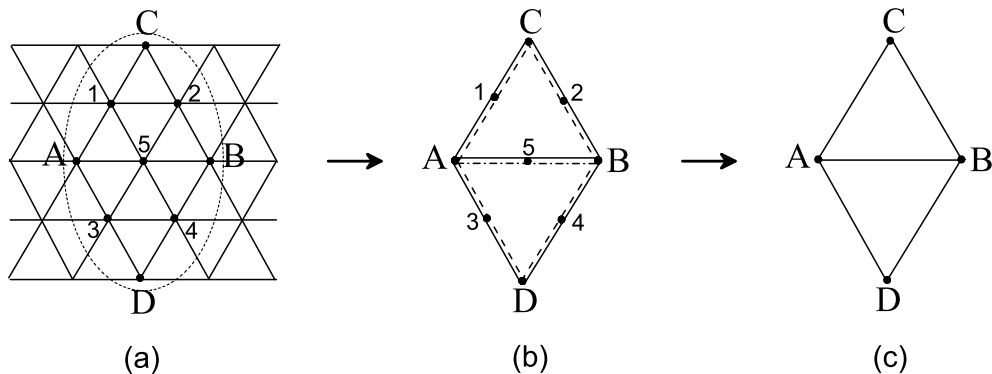


FIG. 1: Traditional bond-moving procedure used on the triangular lattice in previous studies where the sites  $i = 1, 2, \dots, 5$  in (a) and (b) are those to be decimated.

## II. GENERALIZATION

In this section, we will report the improvement we made on the bond-moving procedure by recurring them on the triangular lattice which is a simple but typical configuration with globally symmetry and the spin systems constructed on it can be easily treated with various renormalization methods besides bond-moving.

For comparison, we firstly present the traditional bond-moving procedure generally adopted on the triangular lattice. It proceeds as follows: (1) selecting a cluster of lattice sites as is shown in fig.1a; (2) moving the bonds between each pair of to be eliminated sites  $i = 1, 2, \dots, 5$  to the peripheral position (Seen in fig.1b for an example) to make the lattice coarse-grained; (3) rescaling the system and decimating the to be eliminated sites. Thus it results in a good approximation for the original lattice system and opens up a convenient way for studying the critical properties of the spin systems constructed on it.

However, although it is effective and gives reasonable results for many spin-discrete systems, the traditional bond-moving procedure feels powerless in dealing with the spin-continuous systems such as Gaussian and  $S^4$  models. This is because the cluster selected in step (1) is relatively too large (including five sites to be eliminated) and this results in several complicated integrations (at least five fold of integration) to be treated in the consecutive site-decimating procedure in step (3). Furthermore, in order to keep the globally symmetry unchanged, the moving of the bond between sites 1 and 2 (as well as 3 and 4)

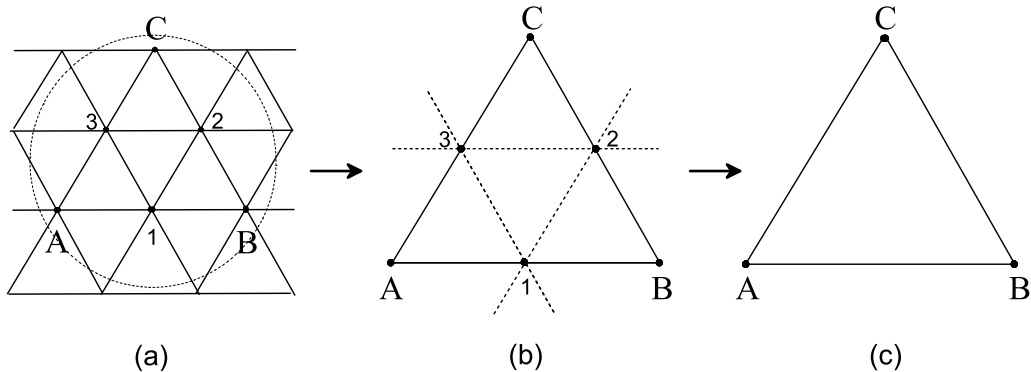


FIG. 2: Improved bond-moving procedure recurring on the triangular lattice where in (a) and (b) the peripheral bonds connecting the three to be eliminated sites 1, 2 and 3 around the selected triplet  $\triangle ABC$  are drawn at half length.

must be operated in a way different from other bonds. This reveals some irregularity of the bond-moving recursion procedure.

Basing on these considerations, we present here an improved bond-moving recursion method to avoid the occurrence of these incongruities. It is proceeded in such a little different way: (1) selecting a cluster of only six lattice sites as a basic unit for recursion where the to be moved peripheral bonds connecting the three to be eliminated sites 1, 2 and 3 around the selected triplet  $\triangle ABC$  are regulated to be in the same order of total length by including two half length bonds as is shown in fig.2b; (2) moving the bonds connecting each pair of to be eliminated sites  $i = 1, 2, 3$  to the peripheral position to make the lattice coarse-grained; (3) rescaling the system and decimating the to be eliminated sites. This recursion procedure can be proved to be a more powerful way bringing with great convenience in the study of spin-continuous systems.

### III. PREDOMINANCE ILLUSTRATION

In order to make a display of its predominance in application, we present here an illustrative study on the critical behavior of the spin-continuous Gaussian model constructed on the triangular-lattice by using of our improved bond-moving recursion.

Let us begin with the classical Gaussian effective Hamiltonian

$$H_{\text{eff}} = \sum_{\langle ij \rangle} K \sigma_i \sigma_j - \frac{b}{2} \sum_i \sigma_i^2, \quad (1)$$

where  $\langle ij \rangle$  in the summation represents a certain nearest-neighbor spin pair;  $K = J/k_B T$  is the reduced interaction with  $K > 0$  denotes the ferromagnetic system;  $b$  is the Gaussian distribution constant;  $k_B$  the Boltzmann constant and  $T$  the thermodynamic temperature. The spins can take any real value between  $(-\infty, +\infty)$  and the probability of finding a given spin between  $\sigma_i$  and  $\sigma_i + d\sigma_i$  is assumed to be the Gaussian-type distribution  $p(\sigma_i)d\sigma_i \propto [\exp -(b/2)\sigma_i^2]d\sigma_i$ .

Fig.3 gives the bond-moving and decimation approach to the Gaussian model on the triangular-lattice performed as Gefen et al did on the fractal [16]. Where the peripheral bonds connecting sites 2 and 3 around the selected triplet are drawn at half length and two types of interactions  $K_e$  and  $K$  together with two types of self-energy  $(-b_e s^2/2)$  and  $(-b s^2/2)$  are assigned respectively for differentiation of the to be and not to be eliminated bonds. For the particular case of triangular lattice the numerical value of  $K_e$  and  $K$  is actually identical as well as that of  $b_e$  and  $b$ . In the bond-moving procedure the two half length bonds are considered acting effectively as a whole one and be moved regularly as other bonds. The decimation procedure for the renormalized bond  $K'$  is

$$\begin{aligned} & \int_{-\infty}^{+\infty} \exp \left[ (K + K_e) (s_a s_1 + s_1 s_b) - \frac{b + b_e}{2} (s_a^2 + s_b^2) - \frac{2b + 2b_e}{2} s_1^2 \right] ds_1 \\ &= \int_{-\infty}^{+\infty} \exp \left[ 2K (s_a s_1 + s_1 s_b) - b (s_a^2 + s_b^2) - 2b s_1^2 \right] ds_1 \\ &= C \exp \left[ K' s'_a s'_b - \frac{b}{2} (s_a'^2 + s_b'^2) \right], \end{aligned} \quad (2)$$

where the relations of  $K = K_e$  as well as  $b = b_e$  are used.

By directly integrating  $s_1$  to decimate the intermediate spins it becomes

$$C \exp \left[ \frac{K^2}{b} s_a s_b - \left( b - \frac{K^2}{2b} \right) (s_a^2 + s_b^2) \right] = C \exp \left[ K' s'_a s'_b - \frac{b}{2} (s_a'^2 + s_b'^2) \right], \quad (3)$$

For the continuity of spin sampling, we can rescale the renormalized spins by

$$s'_a = \xi_a s_a \quad \text{and} \quad s'_b = \xi_b s_b \quad (4)$$

with

$$\xi_a^2 = \xi_b^2 = 2 - \frac{K^2}{b^2}. \quad (5)$$

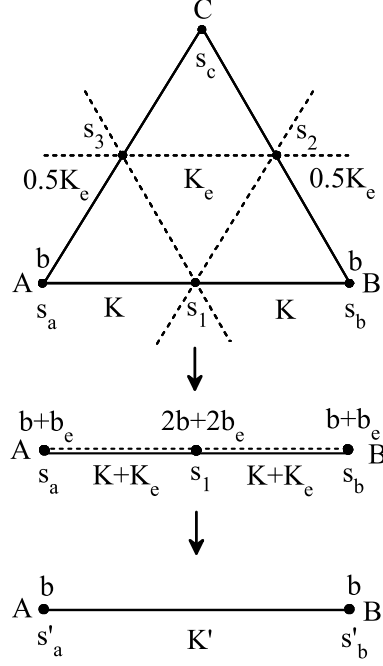


FIG. 3: Bond-moving and decimation procedure of renormalization-group transformation for the renormalized bond  $K'$  between sites A and B of the Gaussian model constructed on the triangular lattice.

Then the recursion relation for  $K'$  reads

$$K' = R(K, K_e) = \frac{1}{\xi_a \xi_b} \frac{K^2}{b} = \frac{K^2 b}{2b^2 - K^2}. \quad (6)$$

The critical point is obtained to be  $K = b$  and the renormalization-group transformation matrix at this point reduced to  $1 \times 1$  with only one eigenvalue

$$\lambda = \frac{\partial K'}{\partial K} \Big|_{K=b} = \frac{4b^3 K}{(2b^2 - K^2)^2} \Big|_{K=b} = 4. \quad (7)$$

Thus we obtain the critical exponent of correlation length

$$\nu = \frac{\ln B}{\ln \lambda} = \frac{\ln 2}{\ln 4} = 0.5 \quad (8)$$

where the triangular lattice is treated as a special kind of fractal with subdividing factor  $B = 2$ . Here we can found that the results we have got is in good conformity with the previous ones [17–19]. However the recursion procedure is quite simple with only one fold of integration need to be performed.

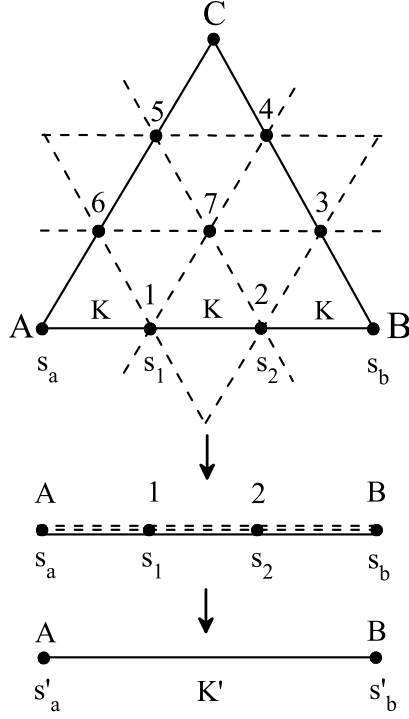


FIG. 4: Bond-moving and decimation procedure for the renormalized bond  $K'$  between sites A and B of the Gaussian model constructed on the triangular lattice in the case of considering the next nearest-neighbor interactions.

This is not the only case to illustrate its predominance in application. In fact, more convenience can be found in treating some more complicated cases. For example, Fig 4 presents the bond-moving and decimation procedure in the case of considering next nearest-neighbor interactions. From which we can see that although the number of to be eliminated sites increases to six, the integration that needs to be performed in the decimation increases only to two folds. This brings little numerical difficulty for the study of spin-continuous models.

Table I reports the related results such as the critical point and correlation length critical exponent of the Gaussian model we obtained in the case of considering two types of long-rang decaying next nearest-neighbor interactions respectively. Where the power law decaying is assumed to be

$$K(r_{mn}) = K \left( \frac{r_{mn}}{a} \right)^{-\alpha}, \quad (9)$$

TABLE I: The related results of the Gaussian model constructed on the triangular lattice in the case of considering next nearest-neighbor interactions. Where  $K_c$  denotes the critical point,  $\lambda$  is the eigenvalue of the transformation matrix,  $\nu$  the correlation length critical exponents of the Gaussian model.

$\alpha$	$K_c^a$	$\lambda^a$	$\nu^a$	$K_c^b$	$\lambda^b$	$\nu^b$
1	0.8828b	6.499	0.5870	0.9123b	6.989	0.5650
2	0.9395b	7.511	0.5449	0.9667b	8.117	0.5246
3	0.9692b	8.178	0.5228	0.9876b	8.651	0.5092
4	0.9845b	8.566	0.5114	0.9954b	8.868	0.5034
5	0.9922b	8.775	0.5058	0.9983b	8.949	0.5013
6	0.9961b	8.887	0.5029	0.9994b	8.984	0.5004
$\rightarrow \infty$	b	9.000	0.5000	b	9.000	0.5000

<sup>a</sup>power law decaying

<sup>b</sup>exponential decaying

while the exponential decaying is supposed to be

$$K(r_{mn}) = K e^{-\left(\frac{r_{mn}}{a} - 1\right)\alpha}, \quad (10)$$

in which  $a$  is the lattice constant,  $\alpha$  the decaying exponent;  $K(r_{mn})$  represents the reduced interaction between spin pairs  $m$  and  $n$  separated by a distance  $r_{mn}$ ; when  $r_{mn} = a$ ,  $K(r_{mn})$  reduced to the nearest neighbor interaction  $K$ .

Seen from Table I again that the results we have got in the case of considering next nearest-neighbor interactions are also in good conformity with the previous studies. But the bond-moving and decimation recursion procedure we have used is quite simple than the usual means reviewed in Sec II. This reveals the improved bond-moving renormalization group transformation procedure we reported in this paper is trustworthy and can bring great convenience in future applications.

#### IV. SUMMARY AND DISCUSSION

In summary, we have presented in this paper an improved Migdal-Kadanoff bond-moving recursion renormalization group transformation procedure including half length operations.



Results obtained by this means such as the correlation length critical exponents of the traditional Gaussian model constructed on the triangular lattices are found to be in good conformity with the classical results from other studies. In particular, the included half length operation is revealed to be able to bring with great conveniences for the study of complicated spin-continuous systems which illustrates a remarkable predominance in application of our generalized Migdal-Kadanoff bond-moving recursions. With these advantages, the future applying of this method on more complicated spin systems such as the  $S^4$  model and even some fractal systems is worth to be eagerly expected.

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- [1] A. A. Migdal, Z. Eksper. Teoret. Fiz. **69** 810 (1975).
  - [2] A. A. Migdal, Z. Eksper. Teoret. Fiz. **69** 1457 (1975).
  - [3] L. P. Kadanoff, Ann. Phys. (N. Y.) **100** 359 (1976).
  - [4] L. P. Kadanoff and A. Houghton, Phys. Rev. B **11** 377 (1975).
  - [5] D. Nelson and M. Fisher, Ann. Phys. (N. Y.) **91** 226 (1975).
  - [6] L. P. Kadanoff, Physics **6** 263 (1966).
  - [7] T. Neimeijer and J. M. J. Van Leeuwen, Physica (Utrecht) **71** 17 (1974).
  - [8] M. Nauenberg and B. Nienhaus, Phys. Rev. Lett. **33** 1598 (1974); Phys. Rev. B **11** 4175 (1975).
  - [9] L. P. Kadanoff, Phys. Rev. Lett. **34** 1005 (1975).
  - [10] L. P. Kadanoff, A. Houghton and M. C. Yalabik, J. Stat. Phys. March 1976.
  - [11] T. L. Bell and K. G. Wilson, Phys. Rev. B **10** 3975 (1974).
  - [12] K. G. Wilson, Phys. Rev. D **10** 2445 (1974).
  - [13] R. Balkan, J. Drouffe and C. Itzykson, Phys. Rev. D **11** 395 (1975).

- [14] H. A. Kramers and G. H. Wannier, Phys. Rev. **60** 252 (1941).
- [15] F. Wegner, J. Math. Phys. **12** 2259 (1971).
- [16] Y. Gefen, A. Aharony and B. B. Mandelbrot, J. Phys. A: Math. Gen. **17** 1277 (1984).
- [17] C. Y. Wang and X. M. Kong, Mod. Phys. Lett. B, **24**, 671 (2010).
- [18] C. Y. Wang and X. M. Kong, Acta. Phys. Sin. **54**, 4365 (2005).
- [19] Z. Q. Lin and X. M. Kong, Chin. Phys. Lett. 18, 882 (2001).